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Variational inclusions problems with applications to Ekeland's variational principle, fixed point and optimization problems

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Abstract In this paper, we prove the existence theorems of two types of systems of variational inclusions problem. From these existence results, we establish Ekeland's variational principle on topological vector space, existence theorems of common fixed point, existence theorems for the semi-infinite problems, mathematical programs with fixed points and equilibrium constraints, and vector mathematical programs with variational inclusions constraints.

Keywords Ekeland's variational principle · Systems of variational inclusions problem · Semi-infinite problem · Mathematical programs with variational inclusions constraints · Common fixed point · Upper (lower) semi-continuous multivalued map

1 Introduction

In 1979, Robinson [24] studied the following parametric variational system:

Given $x \in \mathbb{R}^n$, find y such that

$$0 \in g(x, y) + Q(x, y), \tag{1}$$

where $g: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ is a single valued function and $Q: \mathbb{R}^n \times \mathbb{R}^m \multimap \mathbb{R}^p$ is a multivalued map. It is known that model (1) covers variational inequalities problems and a vast of variational system important in applications. Since then various types of variational inclusions problems have been extended and generalized by Huang [10], etc.

Mordukhovich [23] studied the following problem:

$$\min \varphi(x, y), \quad \text{subject to } y \in S(x), \quad x \in \Omega, \tag{2}$$

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where g and Q are defined as in (1), S: $X \multimap Y$ is given by

$$S(x) = \{y \in Y : 0 \in g(x, y) + Q(x, y)\}$$

and $X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}^m$, and φ: $X \times Y \to \mathbb{R}^s$ is a function. He studied the optimal conditions of this type of problem.

Let $X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}^m$, let f: $X \to \mathbb{R}$, g: $X \to \mathbb{R}$, and h: $X \times Y \to \mathbb{R}$ be functions, H: $X \multimap Y$ be a multivalued map. The semi-infinite problem is the problem:
$\min f(x)$ s.t. $g(x) \geq 0$, and $h(x, v) \geq 0 \quad$ for all $v \in H(x)$.

The semi-infinite problem represents an important class of optimization problem which has been invistigated in a number of papers and books (see, e.g. [2,12,15,18,19,22]) and references therein. As usual, these papers mainly dealed with optimal conditions and develop numerical methods to solve these problems. Typically the existence of feasible solution is tacitly assumed in their work. Therefore it is important to establish the existence theorem of feasible solutions to semi-infinite problems. Recently, Lin et al. [15,18,19] and Lin [12,16] investigated the sufficient conditions for the existence of solution of this type of problem. In some optimization problems the feasible points are the solutions of certain equilibrium problems and fixed points of certain multivalued maps. The recently appeared paper Lin [16] is the first one to study this type of problem.

The celebrated Ekeland's variational principle [4,5] is an important tool in nonlinear analysis. Generalizations and variants were developed, see [7,13] and references there in. Recently Hamel [8] studied the Ekeland's variational principle on sequentially complete locally convex topological vector space (in short t.v.s.), Isac [11] studied vector Ekeland's type variational principle for functions defined on sequentially complete locally convec t.v.s. with values in a Banach space, Wong [26] studied the Ekeland's principle on bornological vector space. Lin and Du [20], Ekeland's variational principle on t.v.s. was established by using an existence theorem of an equilibrium problem.

Let I be an index set. For each $i \in I$, let Z_i be a real t.v.s., X_i and Y_i be nonempty closed convex subsets of locally convex space E_i and V_i, respectively. Let $X = \prod_{i \in I} X_i$ and $Y = \prod_{i \in I} Y_i$. For each $i \in I$, let B_i: $X \times Y \multimap X_i$, S_i: $X \multimap Y_i$, and L_i: $X \times Y \times Y_i \multimap Z_i$ be multivalued maps. Recently, Lin [16] had studied the following type of systems of variational inclusions problems:

(SVIP) Find $\bar{x} = (\bar{x}_i)_{i \in I} \in X$, $\bar{y} = (\bar{y}_i)_{i \in I} \in Y$ such that for each $i \in I$, $\bar{x}_i \in B_i(\bar{x}, \bar{y})$, $\bar{y}_i \in S_i(\bar{x})$, and $0 \in L_i(\bar{x}, \bar{y}, v_i)$ for all $v_i \in S_i(\bar{x})$ (i.e., $0 \in \bigcap_{v_i \in S_i(\bar{x})} L_i(\bar{x}, \bar{y}, v_i)$)

and established the existence theorem of this problem. Use this result, he established the existence theorems of solutions of systems of generalized equations, systems of generalized vector quasi-equilibrium problems, collective variational fixed point, mathematical program with systems of variational inclusions constraints, mathematical program with systems of equilibrium constraints and systems of bilevel problem, and semi-infinite problem with systems of equilibrium constraints.

One easily sees that the above problems also have many connections with the following problems.

Before we state those problems, we introduce notations that will be used throughout this paper unless otherwise specified. For each $i \in I$, let Y_i be a nonempty closed convex subset of a t.v.s. V_i, U_i and Z_i be real t.v.s. Let X be a nonempty subset of a t.v.s. E, $u \in X$ and $Y = \Pi_{i \in I} Y_i$. For each $i \in I$, let F_i: $Y \multimap U_i$, G_i: $Y \times Y_i \multimap Z_i$ and T_i: $Y \multimap Y_i$ be multivalued maps.

In this paper, we study the following type of systems of variational inclusions problems:

(SVIP) Find $\bar{y} \in Y$ such that $0 \in F_i(\bar{y})$, $0 \in G_i(\bar{y}, v_i)$ for all $v_i \in T_i(\bar{y})$ and for all $i \in I$.

From (SVIP) we study the following problems:

(1) Find $\bar{y} \in Y$ such that $0 \in F_i(\bar{y}) + P_i(\bar{y})$, $0 \in Q_i(\bar{y}) + G_i(\bar{y}, v_i)$ for all $v_i \in T_i(\bar{y})$ and for all $i \in I$, where P_i, F_i: $Y \multimap U_i$, Q_i: $Y \multimap Z_i$, and G_i: $Y \times Y_i \multimap Z_i$ be multivalued maps.
(2) Find $\bar{y} \in Y$ such that $\bar{y} \in F_i(\bar{y})$, $\bar{y} \in G_i(\bar{y}, v_i)$ for all $v_i \in T_i(\bar{y})$ and for all $i \in I$.
(3) Find $\bar{y} \in Y$ such that $F_i(\bar{y}) \le 0$, $G_i(\bar{y}, v_i) \ge 0$ for all $v_i \in T_i(\bar{y})$ and for all $i \in I$, where F_i: $Y \to \mathbb{R}$ and G_i: $Y \times Y_i \to \mathbb{R}$ are functions.
The following Ekeland's variational principle on t.v.s. is a particular form of (3);
(4) Find $\bar{x} \in X$ such that
(a) $\varepsilon p(u, \bar{x}) \le f(u) - f(\bar{x})$ and
(b) $\varepsilon p(\bar{x}, v) \ge f(\bar{x}) - f(v)$ for all $v \in T(\bar{x})$,
where, f: $X \to (-\infty, \infty]$, and p: $X \times X \to (-\infty, \infty]$ are functions, $u \in X$, $\varepsilon > 0$ and T: $X \multimap X$ is a multivalued map.
A particular form of (4) is the problem
(5) Find $\bar{x} \in X$ such that
(a) $\varepsilon p(u, \bar{x}) \le f(u) - f(\bar{x})$
(b) $\varepsilon p(\bar{x}, v) \ge f(\bar{x}) - f(v)$ for all $v \in X$.
Let Z_0 be a real t.v.s. ordered by a closed convex cone D in Z_0 and f: $Y \multimap Z_0$. As applications of our results, we study the existence theorems of mathematical programs with variational inclusions constraints (MPVI), semi-infinite problems(SI1 and SI2), and mathematical programs with fixed points and equilibrium constraints (PFIEP):

(MPVI) $\text{Min}_D f(y)$, subject to $y \in Y$ such that for each $i \in I$, $0 \in F_i(y)$, and $0 \in G_i(\bar{y}, v_i)$ for all $v_i \in T_i(y)$.
(SI1) $\text{Min}_D f(y)$, subject to $y \in Y$ such that for each $i \in I$, $G_i(y, v_i) \ge 0$ for all $v_i \in T_i(y)$.
(SI2) $\text{Min}_D f(y)$, subject to $y \in Y$ such that for each $i \in I$, $F_i(y) \le 0$, and $G_i(y, v_i) \ge 0$ for all $v_i \in T_i(y)$.
(MPFTEP) $\text{Min}_D f(y)$, subject to $y \in Y$ such that for each $i \in I$, $y \in F_i(y)$, and $G_i(y, v_i) \ge 0$ for all $v_i \in T_i(y)$.

In this paper, we first establish the existence theorems of systems of generalized quasi-variational inclusions problem, from which we prove the existence of common fixed point theorems for two families of multiavlued maps, Ekeland's variational principle, existence theorems of mathematical programs with variational inclusions constraints, and semi-infinite problems. In this paper, we study Ekeland's variational principle in t.v.s. and our results on Ekeland's variational principle include Lin et al. [20] as special case. Our Ekeland's variational principle on t.v.s. can not be reduced from Theorem 4.2 [26], Theorem 7 [11], and Theorem 2 [8] and vice versa. Our results on mathematical program with variational inclusions constraints, mathematical programs with fixed points and equilibrium constraints and semi-infinite problems are different from Theorems 6.1–6.4 in ref. [10], Corollaries 5.1 and 5.4 in ref. [18], Remark in ref. [12], Theorem 7 in ref. [15], Remark 5.1 in ref. [16].

2 Preliminaries

Let V and W be nonempty sets, a multivalued map $T: V \multimap W$ be a function from V into the power set of W. Let $T: V \multimap W, x \in V, y \in W$, we define $x \in T^{-}(y)$ if and only if $y \in T(x)$. Let V and W be topological spaces (in short t.s.), and let $T: V \multimap W$ be a multivalued map. T is said to be upper semi-continuous (in short u.s.c.) (respectively, lower semi-continuous (in short l.s.c.) at $x \in V$, if for every open set U in W with $T(x) \subseteq U$ (respectively, $T(x) \cap U \neq \emptyset$) there exists an open neighborhood $V(x)$ of x such that $T(x') \subseteq U$ (respectively, $T(x') \cap U \neq \emptyset$) for all $x' \in V(x)$; T is said to be u.s.c. (respectively, l.s.c.) on V if T is u.s.c. (respectively, l.s.c.) at every point of V; T is continuous at x if T is both u.s.c. and l.s.c. at x; T is compact if there exists a compact set $K \subseteq W$ such that $T(V) \subseteq K$; T is closed if $GrT = \{(x,y) \in V \times W : y \in T(x), x \in V\}$ is a closed set in $V \times W$. Let A be a nonempty subset of a vector space E, co$A$ will denote the convex hull of A.

Let Z be a real t.v.s., D a proper closed convex cone in Z. A point $\bar{y} \in A$ is called a vector minimal point of A if for any $y \in A, y - \bar{y} \notin -D \setminus \{0\}$. The set of vector minimal points of A is denoted by $\mathrm{Min}_D A$.

The following lemmas and theorems are needed in this paper.

Lemma 2.1 [25] *Let X and Y be topological spaces, $T: X \multimap Y$ be a multivalued map. Then T is l.s.c. at $x \in X$ if and only if for any $y \in T(x)$ and any net $\{x_\alpha\}_{\alpha \in \Lambda}$ in X converges to x, there exists a net $\{y_\alpha\}_{\alpha \in \Lambda}$, $y_\alpha \in T(x_\alpha)$ for all $\alpha \in A$ with $y_\alpha \to y$, where Λ is an index set.*

Lemma 2.2 [21] *Let Z be a Hausdorff t.v.s., C be a closed convex cone in Z. If A is a nonempty compact subset of Z, then* $\mathrm{Min}_C A \neq \emptyset$.

Theorem 2.1 [1] *Let X and Y be Hausdorff topological spaces, $T: X \multimap Y$ be a multivalued map.*

(1) *If T is an u.s.c. multivalued map with closed values, then T is closed.*
(2) *If Y is a compact space and T is closed, then T is u.s.c.*
(3) *If X is compact and T is an u.s.c. multivalued map with compact values, then $T(X)$ is compact.*

Definition 2.1 Let X be a nonempty convex subset of a vector space E, Y be a nonempty convex subset of a vector space H and Z be a real t.v.s.. Let $F: Y \multimap Z$ and $C: X \multimap Z$ be multivalued maps such that for each $x \in X$, $C(x)$ is a closed convex cone.

(1) F is $C(x)-$ quasi-convex if for any $y_1, y_2 \in Y$ and $\lambda \in [0,1]$, either

$$F(y_1) \subseteq F(\lambda y_1 + (1-\lambda)y_2) + C(x)$$

or

$$F(y_2) \subseteq F(\lambda y_1 + (1-\lambda)y_2) + C(x).$$

(2) F is $\{0\}-$ quasi-convex-like if for any $y_1, y_2 \in Y$ and $\lambda \in [0,1]$, either

$$F(\lambda y_1 + (1-\lambda)y_2) \subseteq F(y_1)$$

or

$$F(\lambda y_1 + (1-\lambda)y_2) \subseteq F(y_2).$$

(3) F is affine if for any $y_1, y_2 \in Y$ and $\lambda \in [0,1]$,

$$F(\lambda y_1 + (1-\lambda)y_2) = \lambda F(y_1) + (1-\lambda)F(y_2).$$

(4) F is concave if for any $y_1, y_2 \in Y$ and $\lambda \in [0,1]$, we have

$$\lambda F(y_1) + (1-\lambda)F(y_2) \subseteq F(\lambda y_1 + (1-\lambda)y_2).$$

Remark 2.1

(a) If $F\colon Y \multimap Z$ is a multivalued map, that F is $C(x)$-quasi-convex does not guarantee that F is $C(x)$-quasi-convex-like.
(b) If $F\colon Y \to Z$ is a function, then F is $C(x)$-quasi-convex if and only if F is $C(x)$-quasi-convex-like.

Theorem 2.2 (Himmelberg [9]) *Let X be a convex subset of a locally convex t.v.s. and D be a nonempty compact subset of X. Suppose $T\colon X \multimap D$ be an u.s.c. multivalued map such that for each $x \in X$, $T(x)$ is a nonempty closed convex subset of D. Then there exists a point $\bar{x} \in X$ such that $\bar{x} \in T(\bar{x})$.*

The following lemma is a special case of Theorem 7 in ref. [3] or Theorem 4.4 in ref. [14].

Lemma 2.3 [3,14] *Let $\{X_i\}_{i\in I}$ be a family of nonempty convex subset, where each X_i is contained in a t.v.s. E_i. For each $i \in I$, let $R_i, S_i\colon X = \Pi_{i\in I}X_i \multimap X_i$ be a multivalued map such that*

(1) *for each $x \in S$, $\mathrm{co}S_i(x) \subset R_i(x)$;*
(2) *for each $x = (x_i)_{i\in I} \in X$, $x_i \notin R_i(x)$;*
(3) *for each $y_i \in X_i$, $S_i^-(y_i)$ is open in X_i;*
(4) *there exist a nonempty compact subset K of X and a nonempty compact convex subset M_i of X_i for all $i \in I$ such that for each $x \in X\backslash K$, there exists $j \in I$ such that $M_j \cap S_j(x) \neq \emptyset$.*

Then there exists $\bar{x} \in X$ such that $S_i(\bar{x}) = \emptyset$ for all $i \in I$.

Throughout this paper, all topological spaces are assumed to be Hausdorff.

3 Existence theorems of variational inclusions problems

The following existence theorem is one of the main results of this paper.

Theorem 3.1 *For each $i \in I$, let $Q_i\colon Y \times Y_i \multimap Z_i$, $B_i, A_i\colon Y \multimap Y_i$ be defined by $A_i(y) = \{v_i \in Y_i : 0 \notin G_i(y, v_i)\}$ and $B_i(y) = \{v_i \in Y_i : 0 \notin Q_i(y, v_i)\}$. For each $i \in I$, suppose that*

(1) *$T_i(Y) \subseteq H_i$, W_i is a closed subset of Y and Y_i is a closed convex subset of V_i, where $H_i = \{y_i \in Y_i : 0 \in F_i(y)$ for $y = (y_i)_{i\in I} \in Y\}$ and $W_i = \{y \in Y : 0 \in F_i(y)\}$;*
(2) *for each $v_i \in Y_i$, $T_i^-(v_i)$ is open;*
(3) *for each $y = (y_i)_{i\in I} \in Y$, $T_i(y)$, and $B_i(y)$ are convex, for each $v_i \in Y_i$, $A_i^-(v_i)$ is open and $0 \in Q_i(y, y_i)$;*
(4) *for each $(y, v_i) \in Y \times Y_i, 0 \notin G_i(y, v_i)$ implies $0 \notin Q_i(y, v_i)$;*

(5) *there exist a nonempty compact subset K of Y and a nonempty compact convex subset M_i of Y_i for each $i \in I$ such that for each $y \in Y \backslash K$, there exists $j \in I$, and $v_j \in M_j \cap T_j(y)$ such that* $0 \notin G_j(y, v_j)$.

Then there exists $\bar{y} \in Y$ such that $0 \in F_i(\bar{y})$ and $0 \in G_i(\bar{y}, v_i)$ for all $v_i \in T_i(\bar{y})$ and for all $i \in I$.

Proof For each $i \in I$, let S_i, R_i: $Y \multimap Y_i$ be defined by

$$S_i(y) = \begin{cases} T_i(y) \cap A_i(y), & \text{if } y \in W_i, \\ T_i(y), & \text{if } y \in Y \backslash W_i \end{cases}$$

and

$$R_i(y) = \begin{cases} T_i(y) \cap B_i(y), & \text{if } y \in W_i, \\ T_i(y), & \text{if } y \in Y \backslash W_i. \end{cases}$$

By (3) and our assumptions, $T_i(y)$, $B_i(y)$, and $R_i(y)$ are convex sets for each $y \in Y$. For each $y = (y_i)_{i \in I} \in Y$, $y_i \notin R_i(y)$. Indeed, if $y \in Y \backslash W_i$, then $0 \notin F_i(y)$ and $y_i \notin H_i$. By(1), $y_i \notin T_i(y)$. Hence $y_i \notin R_i(y)$. By (3), $0 \in Q_i(y, y_i)$, then $y_i \notin B_i(y)$. Therefore if $y \in W_i$, then $y_i \notin T_i(y) \cap B_i(y)$ and $y_i \notin R_i(y)$. It is easy to see that for each $i \in I$ and $v_i \in Y_i$,

$$S_i^-(v_i) = [T_i^-(v_i) \cap A_i^-(v_i)] \cup [(Y \backslash W_i) \cap T_i^-(v_i)].$$

By (1)–(3), $S_i^-(v_i)$ is open for each $i \in I$ and $v_i \in Y_i$. By (4), for each $i \in I$, and $y \in Y$, $A_i(y) \subset B_i(y)$. Hence $\text{co}S_i(y) \subset R_i(y)$. By (5), for each $y \in Y \backslash K$, there exists $j \in I$ such that $M_j \cap S_j(y) \neq \emptyset$. Then, by Lemma 2.3, there exists $\bar{y} \in Y$ such that $S_i(\bar{y}) = \emptyset$ for all $i \in I$. If $\bar{y} \in Y \backslash W_i$, then $S_i(\bar{y}) = T_i(\bar{y}) = \emptyset$. This contradicts with $T_i(y) \neq \emptyset$ for all $y \in Y$. Therefore $\bar{y} \in W$ and $S_i(\bar{y}) = T_i(\bar{y}) \cap A_i(\bar{y}) = \emptyset$. This shows that $0 \in F_i(\bar{y})$ and $0 \in G_i(\bar{y}, v_i)$ for all $v_i \in T_i(\bar{y})$ and for all $i \in I$. □

If we let $G_i = Q_i$ in Theorem 3.1, then we have the following corollary.

Corollary 3.1 *Let I, A_i be the same as in Theorem* 3.1. *Assume that assumptions* (1), (2), *and* (5) *of Theorem* 3.1 *and that condition* (3) *of Theorem 3.1 is replaced by*

(3)′ *for each $y \in Y$, $T_i(y)$ and $A_i(y)$ are convex, for each $v_i \in Y_i$, $A_i^-(v_i)$ is open and* $0 \in G_i(y, y_i)$.

Then there exists $\bar{y} \in Y$ such that $0 \in F_i(\bar{y})$ and $0 \in G_i(\bar{y}, v_i)$ for all $v_i \in T_i(\bar{y})$ and for all $i \in I$.

Remark 3.1

(a) In Theorem 3.1, if for each $i \in I$, Y_i is compact, then condition (5) of Theorem 3.1 can be deleted.
(b) If $F_i(y) = 0$ for all $y \in Y$, then $W_i = Y$, and condition (1) of Theorem 3.1 is satisfied.

Theorem 3.2 *Theorem* 3.1 *is true if for each $i \in I$, Y_i is a closed convex subset of V_i and conditions* (1) *and* (3) *of Theorem 3.1 are replaced by* $(1)_a$ *and* $(3)_a$, *respectively, where*

$(1)_a$ *$T_i(Y) \subseteq H_i$ and $y \multimap F_i(y)$ is closed; and*

(3)$_\mathrm{a}$ *for each* $v_i \in Y_i$, $y \multimap Q_i(y, v_i)$ *is* $\{0\}$*-quasi-convex-like,* $T_i(y)$ *is convex,* $y \multimap G_i(y, v_i)$ *is closed multivalued map and for each* $y = (y_i)_{i\in I} \in Y$, $0 \in Q_i(y, y_i)$.

Proof Take W_i be the same as in Theorem 3.1. By (1)$_\mathrm{a}$, W_i is a closed subset of Y. Indeed, if $y \in \overline{W_i}$, then there exists a net $\{y^\alpha\}_{\alpha\in\Lambda}$ in W_i such that $y^\alpha \to y$. One has $y^\alpha \in Y$ and $0 \in F_i(y^\alpha)$. By (1)$_\mathrm{a}$, $0 \in F_i(y)$. Since Y is a closed set, $y \in Y$. Hence $y \in W_i$ and W_i is a closed set. For each $y \in Y$, $B_i(y)$ is convex. Indeed, if $v_i^1, v_i^2 \in B_i(y)$ and $\lambda \in [0,1]$, then $v_i^1, v_i^2 \in Y_i$, $0 \notin Q_i(y, v_i^1)$ and $0 \notin Q_i(y, v_i^2)$. We want to show that $0 \notin Q_i(y, \lambda v_i^1 + (1-\lambda)v_i^2)$ for all $\lambda \in [0,1]$. Suppose to the contrary that there exists $\lambda_0 \in [0,1]$ such that $0 \in Q_i(y, \lambda_0 v_i^1 + (1-\lambda_0)v_i^2)$. By (3)$_\mathrm{a}$, either $0 \in Q_i(y, \lambda_0 v_i^1 + (1-\lambda_0)v_i^2) \subseteq Q_i(y, v_i^1)$ or $0 \in Q_i(y, \lambda_0 v_i^1 + (1-\lambda_0)v_i^2) \subseteq Q_i(y, v_i^2)$. This leads to a contradiction. Therefore, $0 \notin Q_i(y, \lambda v_i^1 + (1-\lambda)v_i^2)$ for all $\lambda \in [0,1]$. Since Y_i is convex, $\lambda v_i^1 + (1-\lambda)v_i^2 \in Y_i$. Hence $\lambda v_i^1 + (1-\lambda)v_i^2 \in B_i(y)$ for all $\lambda \in [0,1]$ and $B_i(y)$ is convex for each $y \in Y$. For each $v_i \in Y_i$, $A_i^-(v_i)$ is open. Indeed, if $y \in \overline{Y\backslash A_i^-(v_i)}$, then there exists a net $\{y^\alpha\}_{\alpha\in\Lambda}$ in $Y\backslash A_i^-(v_i)$ such that $y^\alpha \to y$. One has $y^\alpha \in Y$ and $0 \in G_i(y^\alpha, v_i)$. We see $y \in Y$. By (3)$_\mathrm{a}$, $0 \in G_i(y, v_i)$. Therefore $y \in Y\backslash A_i^-(v_i)$ and $Y\backslash A_i^-(v_i)$ is closed for each $i \in I$. This shows that $A_i^-(v_i)$ is open for each $v_i \in Y_i$. Then Theorem 3.2 follows from Theorem 3.1. □

The following theorem is equivalent to Theorem 3.1.

Theorem 3.3 *For each* $i \in I$, *let* $P_i\colon Y \multimap U_i$, $L_i\colon Y \multimap Z_i$ *be multivalued maps with nonempty values,* A_i, $T_i\colon Y \multimap Y_i$, *and* $Q_i\colon Y \times Y_i \multimap Z_i$ *be multivalued maps with nonempty convex values. For each* $i \in I$, *suppose that*

(1) $T_i(Y) \subseteq H_i$ *and* W_i *is a closed subset of* Y, *where* $H_i = \{y_i \in Y_i : 0 \in P_i(y) + F_i(y)$ *for* $y = (y_i)_{i\in I} \in Y\}$
and
$W_i = \{y \in Y : 0 \in P_i(y) + F_i(y)\}$;
(2) *for each* $v_i \in Y_i$, $T_i^-(v_i)$ *is open;*
(3) *for each* $y = (y_i)_{i\in I} \in Y$, $B_i(y)$, *and* $T_i(y)$ *are convex, for each* $v_i \in Y_i$, $A_i^-(v_i)$ *is open and* $0 \in Q_i(y, y_i)$, *where*
$A_i(y) = \{v_i \in Y_i : 0 \notin L_i(y) + G_i(y, v_i)\}$
and
$B_i(y) = \{v_i \in Y_i : 0 \notin Q_i(y, v_i)\}$;
(4) *for each* $(y, v_i) \in Y \times Y_i, 0 \notin L_i(y) + G_i(y, v_i)$ *implies* $0 \notin Q_i(y, v_i)$;
(5) *there exist a nonempty compact subset* K *of* Y *and a nonempty compact convex subset* M_i *of* Y_i *for each* $i \in I$ *such that for each* $y \in Y\backslash K$, *there exists* $j \in I$, *and* $v_j \in M_j \cap T_j(y)$ *such that* $0 \notin L_j(y) + G_j(y, v_j)$.

Then there exists $\bar{y} \in Y$ such that $0 \in F_i(\bar{y}) + P_i(\bar{y})$ and $0 \in L_i(\bar{y}) + G_i(\bar{y}, v_i)$ for all $v_i \in T_i(\bar{y})$ and for all $i \in I$.

Theorem 3.4 *Theorem* 3.3 *is true if* y_i *is closed and conditions* (1), (3) *of Theorem* 3.3 *are replaced by* (1)$_\mathrm{b}$ *and* (3)$_\mathrm{b}$, *respectively, where*

(1)$_\mathrm{b}$ $T_i(Y) \subseteq H_i$, $y \multimap P_i(y)$ *is a closed multivalued map and* $y \multimap F_i(y)$ *is an u.s.c. multivalued map with nonempty compact values;*
(3)$_\mathrm{b}$ *for each fixed* $v_i \in Y_i$, $y \multimap Q_i(y, v_i)$ *is* $\{0\}$*-quasi-convex-like, for each* $y = (y_i)_{i\in I}, 0 \in Q_i(y, y_i)$, *and* $T_i(y)$ *is convex;* $y \multimap L_i(y)$ *is closed and* $y \multimap G_i(y, v_i)$ *is an u.s.c. multivalued map with nonempty compact values.*

Proof $y \multimap P_i(y) + F_i(y)$ is closed. Indeed, if $(y, w_i) \in \overline{Gr(P_i + F_i)}$, then there exists a net $(y^\alpha, w_i^\alpha)_{\alpha\in\Lambda} \in Gr(P_i + F_i)$ such that $(y^\alpha, w_i^\alpha) \to (y, w_i)$. We have $w_i^\alpha \in P_i(y^\alpha) + F_i(y^\alpha)$ for all $\alpha \in \Lambda$. There exist $b_i^\alpha \in P_i(y^\alpha)$, $d_i^\alpha \in F_i(y^\alpha)$ such that $w_i^\alpha = b_i^\alpha + d_i^\alpha$. Let $B = \{y^\alpha : \alpha \in \Lambda\} \cup \{y\}$. Then B is compact. By $(1)_\mathrm{b}$ and Theorem 2.1 that $F_i(B) = \cup_{v\in B} F_i(v)$ is compact. Therefore, $\{d_i^\alpha\}_{\alpha\in\Lambda}$ has a subnet $\{d_i^{\alpha_\lambda}\}_{\alpha_\lambda\in\Lambda}$ such that $d_i^{\alpha_\lambda} \to d_i$. Since $y \multimap F_i(y)$ is an u.s.c. multivalued map with closed valued, it follows from Theorem 2.1 that $y \multimap F_i(y)$ is closed. Therefore, $d_i \in F_i(y)$. $b_i^{\alpha\lambda} = w_i^{\alpha\lambda} - d_i^{\alpha\lambda} \to w_i - d_i$. By assumption $(1)_\mathrm{b}$, $w_i - d_i \in P_i(y)$. Hence $w_i \in P_i(y) + d_i \subseteq F_i(y) + P_i(y)$. This shows that $(y, w_i) \in Gr(P_i + F_i)$ and $P_i + F_i$ is closed. Therefore, $y \multimap P_i(y) + F_i(y)$ is closed. Similarly, we can show that $y \multimap L_i(y) + G_i(y, v_i)$ is closed. Then Theorem 3.4 follows from Theorem 3.2. □

Theorem 3.5 *Let Y_i be a nonempty convex subset of a locally convex space V_i. For each $i \in I$, suppose that*

(1) *$T_i\colon Y \multimap Y_i$ is a compact continuous multivalued map with nonempty closed convex values;*
(2) *$(y, v_i) \multimap G_i(y, v_i)$ is a closed multivalued map;*
(3) *for each $y \in Y$, $v_i \multimap G_i(y, v_i)$ is $\{0\}$-quasi-convex-like and $0 \in G_i(y, y_i)$ for all $y = (y_i)_{i\in I} \in Y$ and for each $v_i \in Y_i$, $y \multimap G_i(y, v_i)$ is concave or $\{0\}$-quasi-convex;*
(4) *$y \multimap F_i(y)$ is concave or $\{0\}$-quasi-convex and $\{y_i \in Y_i : 0 \in F_i(y)$ for $y = (y_i)_{i\in I} \in Y\} \neq \emptyset$.*

Then there exists $\bar{y} = (\bar{y}_i)_{i\in I}$ such that $\bar{y}_i \in T_i(\bar{y})$, $0 \in F_i(\bar{y})$, and $0 \in G_i(\bar{y}, v_i)$ for all $v_i \in T_i(\bar{y})$ and for all $i \in I$.

Proof For each $i \in I$, let
$K_i = \{y_i \in Y_i : 0 \in F_i(y)$ for $y = (y_i)_{i\in I} \in Y\}$, and $K = \prod_{i\in I} K_i$.

Then K_i is convex. Indeed, if $y_i^1, y_i^2 \in K_i$ and $\lambda \in [0, 1]$. Let $y^1 = (y_i^1)_{i\in I}$ and $y^2 = (y_i^1)_{i\in I}$, then $y_i^1, y_i^2 \in Y_i$, $y^2 \in Y$, $0 \in F_i(y^1)$, and $0 \in F_i(y^2)$. Since Y_i is convex, $\lambda y_i^1 + (1 - \lambda)y_i^2 \in Y_i$. By (4), it is easy to shows that K_i is a nonempty convex set. For each $i \in I$, let $H_i\colon K \multimap T_i(Y)$ be defined by

$$H_i(y) = \{s_i \in T_i(y) : 0 \in G_i(s, v_i) \text{ for } s = (s_i)_{i\in I} \text{ and for all } v_i \in T_i(y)\}.$$

Follow the same arguments as in Theorem 3.1 in ref. [16], we can show that $H_i\colon K \multimap T_i(Y)$ is a compact u.s.c. multivalued map with nonempty closed convex values. Let $Q\colon K \multimap \prod_{i\in I} T_i(Y)$ be defined by $Q(y) = \prod_{i\in I} H_i(y)$ for $y \in K$. Then it follows from Lemma 3 [6] that $Q\colon K \multimap \prod_{i\in I} T_i(Y)$ is a compact u.s.c. multivalued map with nonempty closed convex values. Then, by Himmelberg fixed point theorem, there exists $\bar{y} = (\bar{y}_i)_{i\in I} \in K$ such that $\bar{y} \in Q_i(\bar{y})$. Then for all $i \in I$, $\bar{y}_i \in T_i(\bar{y})$, $\bar{y}_i \in K_i$, and $0 \in G_i(\bar{y}, v_i)$ for all $v_i \in T_i(\bar{y})$. Since $\bar{y}_i \in K_i$, $0 \in F_i(\bar{y})$. □

Remark 3.2 (a) Theorem 3.5 can not be reduced from Theorem 3.1 [16].

Apply Theorem 3.5 and follow the same argument as in Theorem 3.3, we have the following theorem.

Theorem 3.6 *For each $i \in I$, let Y_i be a nonempty convex subset of a locally convex space V_i. For each $i \in I$, suppose that*

(1) *$T_i\colon Y \multimap Y_i$ is a compact continuous multivalued map with nonempty closed convex values;*

(2) *P_i: $Y \multimap U_i$ is a multivalued map with nonempty values and the set* $J_i = \{y_i \in Y_i : 0 \in P_i(y) + F_i(y)$ *for* $y = (y_i)_{i \in I} \in Y\}$ *is a nonempty convex set;*
(3) *L_i: $Y \multimap Z_i$ is a multivalued map with nonempty values such that for each $y \in Y$, $v_i \multimap L_i(y) + G_i(y, v_i)$ is $\{0\}$-quasi-convex-like and for each $y \in Y$, the set* $H_i(y) = \{s_i \in T_i(y) : 0 \in L_i(s) + G_i(s, v_i)$ *for* $s = (s_i)_{i \in I}$ *and for all* $v_i \in T_i(y)\}$ *is convex and* $0 \in L_i(y) + G_i(y, y_i)$ *for all* $y = (y_i)_{i \in I} \in Y$;
(4) *$(y, v_i) \multimap L_i(y) + G_i(y, v_i)$ is closed.*

Then there exists $\bar{y} = (\bar{y}_i)_{i \in I} \in Y$ such that $\bar{y}_i \in T_i(\bar{y})$, $0 \in P_i(\bar{y}) + F_i(\bar{y})$, $0 \in L_i(\bar{y}) + G_i(\bar{y}, v_i)$ for all $v_i \in T_i(\bar{y})$ and for all $i \in I$.

4 Applications to Ekeland's variational principle and common fixed point theorems

As consequences of Theorems 3.1 and 3.6, we can establish Ekeland's variational principle in t.v.s. and common fixed point theorems.

Theorem 4.1 *For each $i \in I$, let Y_i be closed, F_i: $Y \to \mathbb{R}$, Q_i, G_i: $Y \times Y_i \to \mathbb{R}$ be functions and $H_i = \{y_i \in Y_i : F_i(y) \leq 0$ for $y = (y_i)_{i \in I} \in Y\}$.*

For each $i \in I$, suppose that

(1) *$T_i(Y) \subseteq H_i$ and $y \to F_i(y)$ is a l.s.c. function;*
(2) *for each $v_i \in Y_i$, $T_i^-(v_i)$ is open;*
(3) *for each $y = (y_i)_{i \in I} \in Y$, $T_i(y)$ is convex, $Q_i(y, y_i) \geq 0$, and $v_i \to Q_i(y, v_i)$ is a quasi-convex function and for each $v_i \in Y_i$, $y \to G_i(y, v_i)$ is an u.s.c. function;*
(4) *for each $(y, v_i) \in Y \times Y_i$, $G_i(y, v_i) < 0$ implies $Q_i(y, v_i) < 0$;*
(5) *there exist a nonempty compact subset K of Y and a nonempty compact convex subset M_i of Y_i for each $i \in I$ such that for each $y \in Y \backslash K$, there exist $j \in I$ and $v_j \in M_j \cap T_j(y)$ such that $G_j(y, v_j) < 0$.*

Then there exists $\bar{y} \in Y$ such that $F_i(\bar{y}) \leq 0$, $G_i(\bar{y}, v_i) \geq 0$ for all $v_i \in T_i(\bar{y})$ and for all $i \in I$.

Proof For each $i \in I$, let A_i, B_i: $Y \multimap Y_i$ be defined by

$$A_i(y) = \{v_i \in Y_i : 0 \notin -\mathbb{R}_+ + G_i(y, v_i)\} \text{ and}$$

$$B_i(y) = \{v_i \in Y_i : 0 \notin -\mathbb{R}_+ + G_i(y, v_i)\}.$$

It is easy to see that $H_i = \{y_i \in Y_i : 0 \in \mathbb{R}_+ + F_i(y)$ for $y = (y_i)_{i \in I} \in Y\}$. Let $W_i = \{y \in Y : F_i(y) \leq 0\}$. Then $W_i = \{y \in Y : 0 \in \mathbb{R}_+ + F_i(y)\}$. By (1), W_i is a closed subset of Y. Since $y \to G_i(y, v_i)$ is an u.s.c. function for each $v_i \in Y_i$, $Y \backslash A_i^-(v_i) = \{y \in Y : 0 \in -\mathbb{R}_+ + G_i(y, v_i)\} = \{y \in Y : G_i(y, v_i) \geq 0\}$ is closed. Therefore, $A_i^-(v_i)$ is open for each $v_i \in Y_i$. By (3), $v_i \to Q_i(y, v_i)$ is quasi-convex, then

$$\begin{aligned} B_i(y) &= \{v_i \in Y_i : 0 \notin -\mathbb{R}_+ + Q_i(y, v_i)\} \\ &= \{v_i \in Y_i : Q_i(y, v_i) < 0\} \text{ is convex}. \end{aligned}$$

By (5), for each $y \in Y \backslash K$, there exists $j \in I$ such that $v_j \in M_j \cap T_j(y)$ and $0 \notin -\mathbb{R}_+ + G_j(y, v_j)$.

Then by Theorem 3.1, there exists $\bar{y} \in Y$ such that for each $i \in I$, $0 \in \mathbb{R}_+ + F_i(\bar{y})$ and $0 \in -\mathbb{R}_+ + G_i(\bar{y}, v_i)$ for all $v_i \in T_i(\bar{y})$. That is, $F_i(\bar{y}) \leq 0$ and $G_i(\bar{y}, v_i) \geq 0$ for all $v_i \in T_i(\bar{y})$. □

If we let $G_i = Q_i$ in Theorem 4.1, we have the following corollary.

Corollary 4.1 *Let I, F_i, G_i, H_i and Y_i be the same as in Theorem 4.1. For each $i \in I$, suppose that*

(1) *$T_i(Y) \subset H_i$ and $y \to F_i(y)$ is a l.s.c. function;*
(2) *for each $y = (y_i)_{i \in I} \in Y$, $T_i(y)$ is convex, $G_i(y, y_i) \geq 0$ and $v_i \to G_i(y, v_i)$ is a quasi-convex function and for each $v_i \in Y_i$, $y \to G_i(y, v_i)$ is an u.s.c. function;*
(3) *for each $v_i \in Y_i$, $T_i^-(v_i)$ is open;*
(4) *there exists a nonempty compact subset K of Y and a nonempty compact convex subset M_i of Y_i for each $i \in I$ such that for each $y \in Y \backslash K$, there exist $j \in I$ and $v_j \in M_j \bigcap T_j(y)$ such that $G_j(y, v_j) < 0$.*

Then there exists $\bar{y} \in Y$ such that $F_i(\bar{y}) \leq 0$, and $G_i(\bar{y}, v_i) \geq 0$ for all $v_i \in T_i(\bar{y})$ and for all $i \in I$.

Corollary 4.2 *If we assume assumption* (4) *of Corollary 4.1, and that conditions* (1) *and* (2) *of Corollary 4.1 are replaced by* (1′) *and* (2′)*, respectively, where*

(1′) *$y \to F_i(y)$ is a l.s.c. convex function.*
(2′) *for each $y = (y_i)_{i \in I} \in Y, G_i(y, y_i) \geq 0$ and $v_i \to G_i(y, v_i)$ is a quasi-convex function and for each $v_i \in Y_i, y \to G_i(y, v_i)$ is an u.s.c. function.*

Then there exists $\bar{y} \in Y$ such that $F_i(\bar{y}) \leq 0$ and $G_i(\bar{y}, v_i) \geq 0$ for all $v_i \in H_i$.

Proof Let $T_i(y) = H_i$ for all $y \in Y$. For each $v_i \in Y_i$,

$$T_i^-(v_i) = \begin{cases} Y, & \text{if } v_i \in H_i, \\ \emptyset, & \text{if } v_i \in Y_i \backslash H_i. \end{cases}$$

Therefore $T_i^-(v_i)$ is open and $T_i(y)$ is convex for all $y \in Y$. □

Then Corollary 4.2 follows from Corollary 4.1.

Remark 4.1 If I is singleton, then Corollary 4.2 will be reduced to Theorem 3.3 [20].

Theorem 4.2 *For each $i \in I$, let Y_i be a nonempty convex subset of a locally convex space V_i, F_i: $Y \to \mathbb{R}$ and G_i: $Y \times Y_i \to \mathbb{R}$ be functions. For each $i \in I$, suppose that*

(1) *T_i: $Y \multimap Y_i$ is a compact continuous multivalued map with nonempty closed convex values;*
(2) *$y \to F_i(y)$ is quasi-convex and $\{y_i \in Y_i : F_i(y) \leq 0$ for $y = (y_i)_{i \in I} \in Y\}$ is nonempty;*
(3) *for each $y \in Y$, $v_i \to G_i(y, v_i)$ is quasi-convex; for each $v_i \in Y_i$, $\{s \in Y : G_i(s, v_i) \geq 0\}$ is convex and $G_i(y, y_i) \geq 0$ for all $y = (y_i)_{i \in I} \in Y$; and*
(4) *$y \to G_i(y, v_i)$ is an u.s.c. function for each fixed $v_i \in Y_i$.*

Then there exists $\bar{y} = (\bar{y}_i)_{i \in I} \in Y$ such that $\bar{y}_i \in T_i(\bar{y})$, $F_i(\bar{y}) \leq 0$, $G_i(\bar{y}, v_i) \geq 0$ for all $v_i \in T_i(\bar{y})$ and for all $i \in I$.

Proof Let P_i: $Y \multimap \mathbb{R}$ and L_i: $Y \multimap \mathbb{R}$ be defined by $P_i(y) = \mathbb{R}_+$ and $L_i(y) = -\mathbb{R}_+$ for all $y \in Y$. By (2),

$$\begin{aligned} K_i &= \{y_i \in Y_i, 0 \in \mathbb{R}_+ + F_i(y) \text{ for } y = (y_i)_{i\in I} \in Y\} \\ &= \{y_i \in Y_i : F_i(y) \leq 0 \text{ for } (y_i)_{i\in I} \in Y\} \end{aligned}$$

is a nonempty convex set.

For each $y \in Y$, $v_i \multimap -\mathbb{R}_+ + G_i(y, v_i)$ is $\{0\}$-quasi-convex-like. Indeed, let $v_i^1, v_i^2 \in Y_i$, $\lambda \in [0,1]$, by (3), either
$G_i(y, \lambda v_i^1 + (1-\lambda)v_i^2) \in G_i(y, v_i^1) - \mathbb{R}_+$ or
$G_i(y, \lambda v_i^1 + (1-\lambda)v_i^2) \in G_i(y, v_i^2) - \mathbb{R}_+$.
Therefore, either

$$\begin{aligned} -\mathbb{R}_+ + G_i(y, \lambda v_i^1 + (1-\lambda)v_i^2) &\subseteq G_i(y, v_i^1) - \mathbb{R}_+ - \mathbb{R}_+ \\ &\subseteq G_i(y, v_i^1) - \mathbb{R}_+ \quad \text{or} \\ -\mathbb{R}_+ + G_i(y, \lambda v_i^1 + (1-\lambda)v_i^2) &\subseteq -\mathbb{R}_+ + G_i(y, v_i^2). \end{aligned}$$

This shows that for each $y \in Y$, $v_i \multimap -\mathbb{R}^+ + G_i(y, v_i)$ is $\{0\}$-quasi-convex like. By (3), for each $v_i \in Y_i$, $\{s \in Y : G_i(s, v_i) \geq 0\}$ is convex. Hence for each $y \in Y$, $\{s \in Y : G_i(s, v_i) \geq 0 \text{ for all } v_i \in T_i(y)\} = \cap_{v_i \in T_i(y)}\{s \in Y : G_i(y, v_i) \geq 0\}$ is convex. This shows that for each $y \in Y$,

$$\begin{aligned} H_i(y) &= \{s_i \in T_i(y) : 0 \in -\mathbb{R}_+ + G_i(s, v_i) \text{ for } s = (s_i)_{i\in I} \in Y \text{ and for all } v_i \in T_i(y)\} \\ &= \{s_i \in T_i(y) : G_i(s, v_i) \geq 0 \text{ for } s = (s_i)_{i\in I} \in Y \text{ and for all } v_i \in T_i(y)\} \\ &= T_i(y) \cap \{s_i \in Y_i : G_i(s, v_i) \geq 0 \text{ for } s = (s_i)_{i\in I} \in Y \text{ and for all } v_i \in T_i(y)\} \end{aligned}$$

is convex. $(y, v_i) \multimap -\mathbb{R}_+ + G_i(y, v_i)$ is closed. Indeed, let $J_i(y, v_i) = -\mathbb{R}_+ + G_i(y, v_i)$ and $(y, v_i, a) \in \overline{GrJ_i}$, then there exists a net $(y^\alpha, v_i^\alpha, a^\alpha) \in GrJ_i$ such that $(y^\alpha, v_i^\alpha, a^\alpha) \to (y, v_i, a)$. One has $a^\alpha \in J_i(y^\alpha, v_i^\alpha) = -\mathbb{R}_+ + G_i(y^\alpha, v_i^\alpha)$. Therefore $G_i(y^\alpha, v_i^\alpha) \geq a^\alpha$. By (4), $G_i(y, v_i) \geq \overline{\lim}_{\alpha\to\infty} G_i(y^\alpha, v_i^\alpha) \geq \lim_{\alpha\to\infty} a^\alpha = a$. Hence $a \in -\mathbb{R}_+ + G_i(y, v_i) = J_i(y, v_i)$ and $(y, v_i, a) \in GrJ_i$. This shows that GrJ_i is a closed set and J_i is closed. Therefore $(y, v_i) \multimap -\mathbb{R}_+ + G_i(y, v_i)$ is closed. Then by Theorem 3.6 that there exists $\bar{y} = (\bar{y}_i)_{i\in I} \in Y$ such that $\bar{y}_i \in T_i(\bar{y})$, $0 \in \mathbb{R}_+ + F_i(\bar{y})$ and $0 \in -\mathbb{R}_+ + G_i(\bar{y}, v_i)$ for all $v_i \in T_i(\bar{y})$ and for all $i \in I$. That is $F_i(\bar{y}) \leq 0$ and $G_i(\bar{y}, v_i) \geq 0$ for all $v_i \in T_i(\bar{y})$. □

As consequence of Theorems 4.1 and 4.2, we establish the following existence theorems of Ekeland's variational principle on t.v.s.

Theorem 4.3 *Let X be a nonempty closed convex subset of a t.v.s. E, f: $X \to (-\infty, \infty)$ be a l.s.c. function, $u \in X$ and $\varepsilon > 0$. Let T: $X \multimap X$ be a multivalued map with nonempty convex values, and p, q: $X \times X \to (-\infty, \infty)$ be a function. Suppose that*

1. *$T(X) \subseteq \{y \in X : \varepsilon p(u, y) \leq f(u) - f(y)\}$ and $T^-(v)$ is open for each $v \in X$;*
2. *for each $x \in X$, $q(x, x) \geq 0$ and $v \to q(x, v)$ is quasi-convex;*
3. *for any $x \in X$, $v \to p(x, v)$ is l.s.c.;*
4. *for each $(x, v) \in X \times X$, $\epsilon p(x, v) - f(x) + f(v) < 0$ implies $q(x, v) < 0$;*
5. *for any $v \in X$, $x \to p(x, v)$ is u.s.c.; and*
6. *there exist a nonempty compact subset K of X and a nonempty compact convex subset M of X such that for each $y \in X \backslash K$, there exists $z \in M \cap T(y)$ such that*

$$\varepsilon p(y, z) < f(y) - f(z).$$

Then there exists $\bar{x} \in X$ *such that*

(1) $\varepsilon p(u,\bar{x}) \leq f(u) - f(\bar{x})$;
(2) $\varepsilon p(\bar{x}, v) \geq f(\bar{x}) - f(v)$ *for all* $v \in T(\bar{x})$.

Proof Let $F(x) = \varepsilon p(u,x) - f(u) + f(x)$ and $G(x,v) = \varepsilon p(x,v) - f(x) + f(v)$.
Since $v \to p(u,v)$ and $v \to f(v)$ are l.s.c., $v \to F(v)$ is l.s.c.
By (5), for each $v \in X$, $x \to \varepsilon p(x,v) - f(x) + f(v) = G(x,v)$ is u.s.c.
By (6), for each $y \in X\backslash K$, there exists $z \in M \cap T(y)$ such that $G(y,z) < 0$. Then by Theorem 4.1, there exists $\bar{x} \in X$ such that $\varepsilon p(u,\bar{x}) \leq f(u) - f(\bar{x})$ and $\varepsilon p(\bar{x}, v) \geq f(\bar{x}) - f(v)$ for all $v \in T(\bar{x})$. □

Remark 4.2

(a) If E is a normed linear space, S: $X \to X$ is convex continuous function and p: $X \times X \to \mathbb{R}$ is defined by $p(x,y) = \max\{\|Sx - y\|, \|Sx - Sy\|\}$. Then p satisfies conditions (2), (3), and (5) of Theorem 4.3, but p is not a metric.
(b) Under the assumptions (3) of Theorem 4.3 and f: $X \to (-\infty,\infty)$ is convex, if $T(y) = \{x \in X : \varepsilon p(u,x) \leq f(u) - f(x)\}$ for all $y \in X$. Then $T(y)$ is convex for all $y \in X$ and $T(X) = \{x \in X : \epsilon p(u,x) \leq f(u) - f(x)\}$.
(c) In Theorem 4.3, X is a nonempty closed convex subset of a t.v.s., X need not be a metric space. In Theorem 4.3, f and g are not assumed to have any convexity property.

For the special case of Theorem 4.3, we have the following corollaries.

Corollary 4.3 *Let* X *be a nonempty closed convex subset of a normed linear space* E, f: $X \to (-\infty,\infty)$ *be a l.s.c. function and* q: $X \times X \to (-\infty,\infty)$ *be a function,* $u \in X$ *and* $\varepsilon > 0$. *Let* T: $X \multimap X$ *be a multivalued map with nonempty convex values. Suppose that*

(1) $T(X) \subseteq \{y \in X : \varepsilon\|u - y\| \leq f(u) - f(y)\}$ *and* $T^{-}(v)$ *is open for each* $v \in X$.
(2) *for each* $x \in X$, $q(x,x) \geq 0$ *and* $v \to q(x,v)$ *is quasi-convex;*
(3) *for each* $(x,v) \in X \times X$, $\epsilon\|x - v\| - f(x) - f(v) < 0$ *implies* $q(x,v) < 0$;
(4) *there exist a nonempty compact subset* K *of* X *and a nonempty compact convex subset* M *of* X *such that for each* $y \in X\backslash K$, *there exists* $z \in M \cap T(y)$ *such that* $\varepsilon\|y - z\| < f(y) - f(z)$.

Then there exists $\bar{x} \in X$ *such that*

(1) $\varepsilon\|u - \bar{x}\| \leq f(u) - f(\bar{x})$ *and*
(2) $\varepsilon\|\bar{x} - v\| \geq f(\bar{x}) - f(v)$ *for all* $v \in T(\bar{x})$.

Proof Let $p(x,y) = \|x - y\|$, then Corollary 4.3 follows from Theorem 4.3. □

Remark 4.3 In Corollary 4.3, X is not assumed to be complete.

Corollary 4.4 *Let* X *be a nonempty closed convex subset of a t.v.s.* E, f: $X \to (-\infty,\infty)$ *be a l.s.c. convex function,* $u \in X$ *and* $\varepsilon > 0$. *Let* T: $X \multimap X$ *be a multivalued map with nonempty convex values, and* p: $X \times X \to (-\infty,\infty)$ *be a function. Suppose that*

(1) $T(X) \subseteq \{y \in X : \varepsilon p(u,y) \leq f(u) - f(y)\}$ *and* $T^{-}(v)$ *is open for each* $v \in X$;
(2) $p(x,x) \geq 0$ *for all* $x \in X$;

(3) *for any* $x \in X$, $v \to p(x,v)$ *is convex and l.s.c.;*
(4) *for any* $v \in X$, $x \to p(x,v)$ *is u.s.c.; and*
(5) *there exist a nonempty compact subset* K *of* X *and a nonempty compact convex subset* M *of* X *such that for each* $y \in X\backslash K$, *there exists* $z \in M \cap T(y)$ *such that*

$$\varepsilon p(y,z) < f(y) - f(z).$$

Then there exists $\bar{x} \in X$ *such that*

(1) $\varepsilon p(u,\bar{x}) \le f(u) - f(\bar{x})$;
(2) $\varepsilon p(\bar{x},v) \ge f(\bar{x}) - f(v)$ *for all* $v \in T(\bar{x})$.

Proof Let q: $X \times X \to (-\infty,\infty)$ be defined by $q(x,v) = \epsilon p(x,v) - f(x) + f(v)$. Then Corollary 4.4 follows from Theorem 4.3. □

The following Ekeland's variational principle theorem follows immediately from Corollary 4.4 and the argument as in ref. [20].

Theorem 4.4 [20] *Let* X *be closed subset of a t.v.s.,* $u \in X$ *and* $\varepsilon > 0$. *Let* f: $X \to (-\infty,\infty)$ *be a l.s.c. convex function and* p: $X \times X \to (-\infty,\infty)$ *be a function. Suppose that*

(1) $p(x,x) \ge 0$ *for all* $x \in X$ *and* $p(u,u) = 0$
(2) $p(x,z) \le p(x,y) + p(y,z)$ *for any* $x,y,z \in X$;
(3) *for any* $x \in X$, $p(x,\cdot)$ *is convex and l.s.c.;*
(4) *for any* $y \in X$, $p(\cdot,y)$ *is u.s.c.;*
(5) *there exist a nonempty compact subset* K *of* X *and a nonempty compact convex subset* M *of* X *such that for each* $y \in X\backslash K$, *there exists* $z \in M$ *such that*

$$\varepsilon p(y,z) < f(y) - f(z) \text{ and } \varepsilon p(u,z) \le f(u) - f(z).$$

Then there exists $\bar{x} \in X$ *such that*

(1) $p(u,\bar{x}) \le f(u) - f(\bar{x})$ *and*
(2) $\varepsilon p(\bar{x},v) \ge f(\bar{x}) - f(v)$ *for all* $v \in X$.

Proof Let $W = \{x \in X : \varepsilon p(u,x) \le f(u) - f(x)\}$. Since $x \to \varepsilon p(u,x)$ and $x \to f(x)$ are l.s.c. convex functions, $x \to \varepsilon p(u,x) + f(x) - f(u)$ is a l.s.c. convex function and W is a closed convex subset of X. Let T: $X \multimap X$ be defined by $T(y) = W$ for all $y \in X$. Then

$$T^{-}(z) = \begin{cases} X, & \text{if } z \in W, \\ \emptyset, & \text{if } z \in X\backslash W. \end{cases}$$

Then $T^{-}(z)$ is open for all $z \in X$ and

$$T(X) = W = \{x \in X : \varepsilon p(u,x) \le f(u) - f(x)\}.$$

Then by Theorem 4.3 that there exists $\bar{x} \in X$ such that

(1) $\varepsilon p(u,\bar{x}) \le f(u) - f(\bar{x})$ and
(2) $\varepsilon p(\bar{x},v) \ge f(\bar{x}) - f(v)$ for all $v \in T(\bar{x}) = W$.

If $v \in X\backslash W$, then

$$\begin{aligned} \varepsilon[p(u,\bar{x}) + p(\bar{x},v)] &\ge \varepsilon p(u,v) > f(u) - f(v) \\ &\ge \varepsilon p(u,\bar{x}) + f(\bar{x}) - f(v). \end{aligned}$$

Therefore $\varepsilon p(\bar{x}, v) > f(\bar{x}) - f(v)$ for all $v \in X \backslash W$. Hence $\varepsilon p(\bar{x}, v) \geq f(\bar{x}) - f(v)$ for all $v \in X$. □

The following corollary follows from Theorem 4.4.

Corollary 4.5 *Let X be a closed subset of a metrizable t.v.s. with topology induced by a metric d. Let $f: X \to (-\infty, \infty)$ be a l.s.c. convex function, $u \in X$ and $\varepsilon > 0$. Suppose that*

(1) *for any $x \in X$, $v \to d(x, v)$ is convex;*
(2) *there exist a nonempty compact subset K of X and a nonempty compact convex subset M of X such that for each $y \in X \backslash K$, there exists $z \in M$ such that*

$$\varepsilon d(y, z) < f(y) - f(z) \text{ and } \varepsilon d(u, z) \leq f(u) - f(z).$$

Then there exists $\bar{x} \in X$ such that

(1) $\varepsilon d(u, \bar{x}) \leq f(u) - f(\bar{x})$ *and*
(2) $\varepsilon d(\bar{x}, v) \geq f(\bar{x}) - f(v)$ *for all $v \in X$.*

Remark 4.4 In Corollary 4.5, (X, d) is not assumed to be complete, If X is compact, then condition (2) in Corollary 4.5. can be deleted.

Apply Theorem 4.2 and follow the same argument as in Theorem 4.3, we obtain another version of Ekeland's variational principle.

Theorem 4.5 *Let X be a nonempty convex subset of a locally convex space E, $\varepsilon > 0$ and $u \in X$, $f: X \to (-\infty, \infty)$ be a l.s.c. convex function, $p: X \times X \to (-\infty, \infty)$ be a function. Suppose that*

(1) *$T: X \multimap X$ is a compact continuous multivalued map with nonempty closed convex values;*
(2) *$v \to p(u, v)$ is convex;*
(3) *for each $x \in X$, $p(x, x) \geq 0$ and for each $v \in X$, $x \to p(x, v)$ is an u.s.c. concave function.*

Then there exists $\bar{x} \in X$ such that $\bar{x} \in T(\bar{x})$,

(1) $\varepsilon p(u, \bar{x}) \leq f(u) - f(\bar{x})$ *and*
(2) $\varepsilon p(\bar{x}, v) \geq f(\bar{x}) - f(v)$ *for all $v \in T(\bar{x})$.*

Remark 4.6

(a) If $(X, \|\cdot\|)$ is a normed space and $p: X \times X \to (-\infty, \infty]$ be defined by $p(x, v) = \|v\| - \|x\|$, then p satisfies conditions (2)–(4) of Corollary 4.4 and conditions (2) and (3) of Theorem 4.5, but p is not a metric.
(b) The Ekeland's variational principle in Theorems 4.4–4.6 requires certain convexity assumptions on p and f, but in Theorem 4.3, we do not assume any convexity assumption on p and f. In Corollary 4.3, we do not assume any convexity assumption of f.

Remark 4.7 If we take $p = f \equiv 0$, then Theorem 4.5 will be reduced to Himmelberg fixed point theorem. In fact, these two theorems are equivalent.

For the special case of Theorem 4.5, we establish a common solutions of fixed point and optimization problem.

Corollary 4.6 *Let X be a nonempty convex subset of a locally convex space E, f: $X \to (-\infty, \infty)$ be a l.s.c, convex function , and $u \in X$. Suppose that*

(1) *T: $X \multimap X$ is a compact continuous multivalued map with nonempty closed convex values.*

Then there exists $\bar{x} \in X$ such that $\bar{x} \in T(\bar{x}), f(\bar{x}) \leq f(u), f(v) \geq f(\bar{x})$ for all $v \in T(\bar{x})$.

Proof Let $p(x,y) = 0$ for all $(x,y) \in X \times X$. Then Corollary 4.6 follows from Theorem 4.5. □

As consequences of Theorems 3.3 and 3.6, we establish the following common fixed point theorems.

Theorem 4.6 *For each $i \in I$, let Y_i be a nonempty closed convex subset of t.v.s. V_i for each $i \in I$, let F_i: $Y \multimap Y$, G_i: $Y \times Y_i \multimap Y$ be multivalued maps with nonempty values,*

$$H_i = \{y_i \in Y_i : y \in F_i(y) \text{ for } y = (y_i)_{i \in I} \in Y\}.$$

For each $i \in I$, suppose that

(1) *$T_i(Y) \subseteq H_i$ and $F_i : Y \multimap Y$ is a closed multivalued map with nonempty values;*
(2) *for each $v_i \in Y_i$, $T_i^-(v_i)$ is open and for each $y = (y_i)_{i \in I} \in Y$, $T_i(y)$ is convex and $y \in G_i(y, y_i)$;*
(3) *for each $v_i \in Y_i$, $y \multimap G_i(y, v_i)$ is a closed, $\{0\}$-quasi-convex-like multivalued map; and*
(4) *there exist a nonempty compact subset K of Y and a nonempty compact convex subset M_i of Y_i for each $i \in I$ such that for each $y \in Y \backslash K$, there exists $j \in I$ and $v_j \in M_j \cap T_j(y)$ such that $0 \notin -y + G_j(y, v_j)$.*

Then there exists $\bar{y} \in Y$ such that $\bar{y} \in F_i(\bar{y}), \bar{y} \in G_i(\bar{y}, v_i)$ for all $v_i \in T_i(\bar{y})$ and for all $i \in I$.

Proof For each $i \in I$, let $P_i(y) = \{-y\}$, $L_i(y) = \{-y\}$. Then Theorem 4.6 follows with the same argument as in Theorem 3.3. □

Theorem 4.7 *For each $i \in I$, let Y_i be a nonempty convex subset of a locally convex space V_i, F_i: $Y \multimap Y$, G_i: $X \times Y_i \multimap Y$ be multivalued maps with nonempty values. For each $i \in I$, suppose that*

(1) *T_i: $Y \multimap Y_i$ is a compact continuous multivalued map with nonempty closed convex values;*
(2) *$y \multimap F_i(y)$ is a concave multivalued map and $\{y_i \in Y_i : y \in F_i(y)$ for $y = (y_i)_{i \in I} \in Y\} \neq \emptyset$*
(3) *for each $y = (y_i)_{i \in I} \in Y$, $v_i \multimap G_i(y, v_i)$ is $\{0\}$-quasi-convex-like and $y \in G_i(y, y_i)$ and for each $v_i \in Y_i$, $y \multimap G_i(y, v_i)$ is a concave multivalued map;*
(4) *$G_i : Y \times Y_i \multimap Y_i$ is closed.*

Then there exists $\bar{y} = (\bar{y}_i)_{i \in I} \in Y$ such that $\bar{y}_i \in T_i(\bar{y}), \bar{y} \in F_i(\bar{y}), \bar{y} \in G_i(\bar{y}, v_i)$ for all $v_i \in T_i(\bar{y})$ and for all $i \in I$.

Proof Let $P_i(y) = \{-y\}$, $L_i(y) = \{-y\}$. Since $y \multimap F_i(y)$ is concave, it is easy to see that

$$K_i = \{y_i \in Y_i : 0 \in -y + F_i(y) \text{ for } y = (y_i)_{i\in I} \in Y\}$$
$$= \{y_i \in Y_i : y \in F_i(y) \text{ for } y = (y_i)_{i\in I} \in Y\}$$

is a convex set. By (2), $K_i \neq \emptyset$. It is easy to see that set

$$H_i(y) = \{s_i \in T_i(y) : 0 \in -s + G_i(s, v_i) \text{ for } s = (s_i)_{i\in I} \in Y \text{ and for all } v_i \in T_i(y)\}$$
$$= \{s_i \in T_i(y) : s \in G_i(s, v_i) \text{ for } s = (s_i)_{i\in I} \in Y \text{ and for all } v_i \in T_i(y)\}.$$

Since $y \multimap G_i(y, v_i)$ is concave.

$$H_i(y) = \cap_{v_i\in T_i(y)}\{s_i \in T_i(y) : s \in G_i(s, v_i) \text{ for } s = (s_i)_{i\in I} \in Y\} \text{ is convex.}$$

Since $y \multimap G_i(y, v_i)$ is closed for each $v_i \in Y_i$ and T_i is closed, it is easy to show that H_i is closed. Then follow the same argument as in Theorem 3.5, we can prove Theorem 4.7. □

As a consequence of Theorem 4.7, we obtain another common fixed point for two families of multivlaued maps. This fixed point theorem contains Himmelberg fixed point theorem as special case.

Corollary 4.7 *For each $i \in I$, let Y_i be a nonempty convex subset of a locally convex space V_i, F_i: $Y \multimap Y$, T_i: $Y \multimap Y_i$, be multivalued maps with nonempty values. For each $i \in I$, suppose that*

(1) *T_i: $Y \multimap Y_i$ is a compact u.s.c. multivalued map with nonempty closed convex values;*
(2) *$y \multimap F_i(y)$ is a concave multivalued map and $\{y_i \in Y_i : y \in F_i(y)$ for $y = (y_i)_{i\in I} \in Y\} \neq \emptyset$.*

Then there exists $\bar{y} = (\bar{y}_i)_{i\in I} \in Y$ such that $\bar{y}_i \in T_i(\bar{y})$, $\bar{y} \in F_i(\bar{y})$ for all $i \in I$.

Proof Let $G_i(y, v_i) = y$ for all $(y, v_i) \in Y \times Y_i$. Then Corollary 4.7 follows from Theorem 4.7. □

Remark 4.8 (1) If I is a singleton and we let $F(y) = y$ for all $y \in Y$, then Corollary 4.7 will be reduced to Himmelberg fixed point theorem. As Theorem 3.5 follows from Himmelberg fixed point theorem and Himmelberg fixed point theorem is a special case of Theorem 3.5, we see that Theorem 3.5 and Himmelberg's fixed point theorem are equivalent.

5 Existence theorems of mathematical programs with variational inclusions constraints and semi-infinite problems

In this section, we first study the following mathematical program with systems of variational inclusions constraints.

Theorem 5.1 *In Theorem 3.2, if we assume further that f: $Y \multimap Z_0$ is an u.s.c. multivalued map with nonempty compact values, where Z_0 is a real t.v.s ordered by a proper closed convex cone D. Then there exists a solution to the problem:*

(MPVI) $\mathrm{Min}_D f(y)$ *such that* $y \in Y$.
$0 \in F_i(y), 0 \in G_i(y, v_i)$ *for all* $v_i \in T_i(y)$ *and for all* $i \in I$.

Proof Let $B_i = \{y \in Y : 0 \in F_i(y)$ and $0 \in G_i(y, v_i)$ for all $v_i \in T_i(y)\}$ and $B = \cap_{i \in I} B_i$. By Theorem 3.2 that there exists $\bar{y} \in Y$ such that for each $i \in I$, $0 \in F_i(\bar{y})$, and $0 \in G_i(\bar{y}, v_i)$ for all $v_i \in T_i(\bar{y})$. Therefore $\bar{y} \in B \neq \emptyset$. By condition (4) of Theorem 3.2 that $\bar{y} \in K$ and $B \subseteq K$. B_i is closed for each $i \in I$. Indeed, if $y \in \overline{B_i}$, then there exists a net $\{y^\alpha\}_{\alpha \in \Lambda}$ in B_i such that $y^\alpha \to y$. One has $y^\alpha \to y$ and $0 \in F_i(y^\alpha)$, $0 \in G_i(y^\alpha, v_i)$ for all $v_i \in T_i(y^\alpha)$. Let $v_i \in T_i(y)$. By (ii), T_i is l.s.c. By Lemma 2.1 that there exists a net $\{v_i^\alpha\}$ such that $v_i^\alpha \in T_i(y^\alpha)$ and $v_i^\alpha \to v_i$. Therefore $0 \in G_i(y^\alpha, v_i^\alpha)$. By assumption, $y \multimap F_i(y)$ and $(y, v_i) \multimap G_i(y, v_i)$ are closed. $0 \in F_i(y)$ and $0 \in G_i(y, v_i)$. Since Y is closed, $y \in Y$. Therefore $y \in B_i$ and B_i is closed. Hence $B = \cap_{i \in I} B_i$ is closed. But $B \subseteq K$ and K is compact. B is a compact set. Since f is an u.s.c. multivalued map with compact values, it follows from Theorem 2.1 that $f(B)$ is compact. Then Theorem 5.1 follows from Lemma 2.2. □

Remark 5.1 Theorem 5.1 is true if the condition that "f: $Y \multimap Z_0$ is an u.s.c. multivalued map with nonempty compact values" is replaced by "f: $Y \to \mathbb{R}$ is a l.s.c. function."

Proof Let B be defined as in Theorem 5.1, we see in the Proof of Theorem 5.1 that B is compact. Since f: $Y \to \mathbb{R}$ is l.s.c., there exists a solution to (MPVI).

Theorem 4.1 can be used to prove an existence theorem of the following semi-infinite problem.

(SI2) $\mathrm{Min}_D f(y)$ subject to $y \in Y$ such that for each $i \in I$, $F_i(y) \leq 0$ and $G_i(y, v_i) \geq 0$ for all $v_i \in T_i(y)$.

Theorem 5.2 *In Theorem* 4.1*, if we assume further that* f: $Y \multimap Z_0$ *is an u.s.c. multivalued map with nonempty compact values and* Z_0 *and* D *are defined as in Theorem 5.1. Then there exists a solution to the problem* (SI2).

Proof Let $B_i = \{y \in Y : F_i(y) \leq 0$ and $G_i(y, v_i) \geq 0$ for all $v_i \in T_i(y)\}$ and $B = \cap_{i \in I} B_i$. By Theorem 4.1, $B \neq \emptyset$. By condition (4) of Theorem 4.1, $B \subseteq K$. For each $i \in I$, B_i is closed. Indeed, if $y \in \bar{B}_i$, then there exists a net $\{y^\alpha\}_{\alpha \in \Lambda}$ in B_i such that $y^\alpha \to y$. One has $y^\alpha \in Y$, $F_i(y^\alpha) \leq 0$ and $G_i(y^\alpha, v_i) \geq 0$ for all $v_i \in T_i(y^\alpha)$. Let $v_i \in T_i(y)$. By (2), T_i is l.s.c., there exists a net $\{v_i^\alpha\}_{\alpha \in \Lambda}$ in $T_i(y^\alpha)$ such that $v_i^\alpha \to v_i$. Hence $G_i(y^\alpha, v_i^\alpha) \geq 0$. Since F_i is l.s.c., G_i is u.s.c. and Y is closed, $y \in Y$, $F_i(y) \leq 0$ and $G_i(y, v_i) \geq 0$ for all $v_i \in T_i(y)$. This shows that B is closed. Since $B \subseteq K$ and K is compact. B is compact. Follow the same argument as in Theorem 5.1, we can prove Theorem 5.2. □

Remark 5.2

(a) In Theorem 5.2, if we assume that f: $Y \to \mathbb{R}$ is a l.s.c. function, then there exists a solution to the problem:
min $f(y)$ subject to $y \in Y$ such that for each $i \in I$, $F_i(y) \leq 0$ and $G_i(y, v_i) \geq 0$ for all $v_i \in T_i(y)$.

(b) In Theorem 5.2, if H_i: $Y \to Y^*$ is a continuous function and η_i: $Y_i \times Y_i \to Y_i$ is an affine continuous function such that $\eta_i(y_i, y_i) \geq 0$ for all $y_i \in Y_i$. Let $\langle \cdot, \cdot \rangle$ be the dual pair between Y_i and Y_i^*. Then it follows from Theorem 5.2 that there exists a solution to the problem:
$\mathrm{Min}_D f(y)$ subject to $y = (y_i)_{i \in I} \in Y$ such that for each $i \in I$, $F_i(y) \leq 0$, $\langle H_i(y), \eta_i(y_i, v_i) \rangle \geq 0$ for all $v_i \in T_i(y)$.

Apply Theorem 4.2 and follow the same argument as in Theorem 5.2, we have the following result.

Theorem 5.3 *In Theorem* 4.2, *if we assume further that* $f\colon Y \multimap Z_0$ *is an u.s.c. multivalued map with nonempty compact values and* Z_0 *and* D *are defined as in Theorem* 5.1. *Then there exists a solution to the problem:*

(MPFPEP) $\mathrm{Min}_D f(y)$ *subject to* $y = (y_i)_{i\in I} \in Y$ *such that for each* $i \in I$, $y_i \in T_i(y)$, $F_i(y) \leq 0$ *and* $G_i(y, v_i) \geq 0$ *for all* $v_i \in T_i(y)$.

If we apply Theorem 3.3 and follow the same argument as in Theorems 4.1 and 5.1, we have the following result.

Theorem 5.4 *In Theorem* 4.1, *if*

$$H_i = \{y_i \in Y_i : F_i(y) \geq 0 \text{ for } y = (y_i)_{i\in I} \in Y\}$$

is replaced by

$$H_i' = \{y_i \in Y_i : 0 \in F_i(y) \text{ for } y = (y_i)_{i\in I} \in Y\}$$

condition (1) *is replaced by* (1′) *and assume further that* $f : Y \multimap Z_0$ *is an u.s.c. multivalued map with nonempty compact values and* Z_0 *and* D *are the same as in Theorem* 5.1, *where*

(1′) $T_i(Y) \subset H_i'$ *and* $F_i\colon Y \to Y$ *is a closed multivalued map.*

Then there exists a solution to the problem:

(MPVIEP) $\mathrm{Min}_D f(y)$ *subject to* $y \in Y$ *such that for each* $i \in I$, $0 \in F_i(y)$ *and* $G_i(y, v_i) \geq 0$ *for all* $v_i \in T_i(y)$.

Remark 5.4 In Theorem 5.4, if

$$H_i' = \{y_i \in Y_i : 0 \in F_i(y) \text{ for } y = (y_i)_{i\in I} \in Y\}$$

is replaced by

$$H_i'' = \{y_i \in Y_i : y \in F_i(y) \text{ for } y = (y_i)_{i\in I} \in Y\}.$$

Then there exists a solution to the problem:

(MPFPEP) $\mathrm{Min}_D f(y)$ subject to $y \in Y$ such that for each $i \in I$, $y \in F_i(y)$ and $G_i(y, v_i) \geq 0$ for all $v_i \in T_i(y)$.

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